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Global relationships between two-dimensional water wave potentials

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When a body interacts with small-amplitude surface waves in an ideal fluid, the resulting velocity potential may be split into a part due to the scattering of waves by the fixed body and a part due to the radiation of waves by the moving body into otherwise calm water. A formula is derived which expresses the two-dimensional scattering potential in terms of the heave and sway radiation potentials at all points in the fluid. This result generalizes known reciprocity relations which express quantities such as the exciting forces in terms of the amplitudes of the radiated waves. To illustrate the use of this formula beyond the reciprocity relations, equations are derived which relate higher-order scattering and radiation forces. In addition, an expression for the scattering potential due to a wave incident from one infinity in terms of the scattering potential due to a wave from the other infinity is obtained.

1. Introduction

The linear analysis of the interaction of water waves with fixed and floating structures is a well-established area of research. Exact, analytic representations for the diffracted wave field are rare and so a substantial amount of effort has been invested in the development of accurate and efficient numerical codes such as described in the review by Yeung (1982). However, a few analytic solutions do exist and perhaps the most notable is the solution given in MacCamy & Fuchs (1954) for the diffracted wave field around a vertical, circular cylinder which extends throughout the whole fluid depth. Another exact solution is available for a two-dimensional geometry, the finite surface-piercing vertical plate. This was derived by Ursell (1947) and he obtained simple, explicit expressions for the reflection and transmission coefficients for this geometry. His work was extended by Evans (1970) who derived formulae for the reflection and transmission coefficients for the finite submerged vertical plate. Further exact expressions for scattering coefficients were found by Heins (1950) who considered the diffraction of waves obliquely incident on a semi-infinite horizontal plate. John (1948) also obtained an explicit solution for the scattering of waves by a surface-piercing barrier which is inclined at an angle $\pi/2n$ to the horizontal. However, the solution rapidly becomes very complicated as n increases and the only example he looked at in detail was the case $n = 1$ which corresponds to the vertical barrier. For more general geometries, analytic solutions do not exist although analytical approximations may often be obtained if some additional assumption is made, such as the frequency being high or low.

The exact and approximate solutions are, of course, useful in their own right but they also provide a valuable check on the accuracy of any numerical code. Further checks on numerical work come from the so-called 'reciprocity relations' which relate various hydrodynamic quantities. If a body moves with zero forward speed, the assumption of linear theory allows the velocity potential to be split into a part due to the scattering of waves by a fixed body and a part due to the radiation of waves by the oscillating body into otherwise calm water. The radiation potential may be further split into separate potentials, each of which is due to the body moving in a different mode of motion. The scattering and radiation potentials are not all independent and, for example, Haskind (1957) related the amplitude of the far-field radiated waves to the force on the body due to the scattering potential. Other such reciprocity relations exist and have been obtained over a period of years, but a systematic derivation of all the first-order relations from Green's theorem was finally made by Newman (1976).

These results have been extended to yield expressions for some of the higher-order forces on the body. For example, the mean horizontal drift force and vertical moment on a body were expressed in terms of the far-field scattered waves by Maruo (1960) and Newman (1967) respectively. A similar representation for the component of the double-frequency force due to the first-order potential was obtained by McIver (1992). All the results derived, however, relate far-field quantities or integrated near-field quantities but do not relate the potentials at arbitrary points in the fluid. The purpose of this work is to show how, in two dimensions, the first-order scattering potential may be written in terms of the radiation potentials at every point in the fluid. Thus, it is not necessary to solve directly for the scattering potential at all, even if it is required at an arbitrary point in the fluid. An example of where the potential is needed at several different points is in the calculation of the forcing term which generates the second-order scattering potential and which involves products of the first-order scattering potential over the whole free surface. By using the formulae derived here, this forcing term may be written directly in terms of the first-order radiation potentials which would lead to numerical savings if these terms had already been calculated in order to solve for the second-order radiation potentials. In addition, equations relating the components of the second-order scattering and radiation forces due to the first-order potential may be easily obtained.

The relations between the first-order scattering and radiation potentials are derived in §2. The method used is to form a combination of the radiation potentials which has the same far-field behaviour as the scattering potential and satisfies the same body boundary conditions. It is then shown that, provided the potential is unique, this combination must be equal to the scattering potential. In §3 a similar analysis is used to relate the scattering potential due to a wave incident from one infinity to that due to a wave incident from the other infinity at every point in the fluid. In particular, relations between the potential at pairs of points in the fluid are derived for symmetric bodies. Finally, in §4, relations between the components of the vertical second-order scattering and radiation forces due to the first-order potential are derived.

2. The scattering and radiation potentials

A two-dimensional body is either partially or totally immersed in an ideal fluid of depth h, as illustrated in figure 1. Cartesian axes are chosen so that the origin is in the mean free surface and the z-axis points vertically upwards. The motion is assumed to be oscillatory with angular frequency ω and to be sufficiently small that linear theory may be used. Thus, the velocity potential may be split into the sum of a potential due to the scattering of waves by the fixed body and potentials due to the radiation of waves into otherwise calm water by the body moving in one of its modes

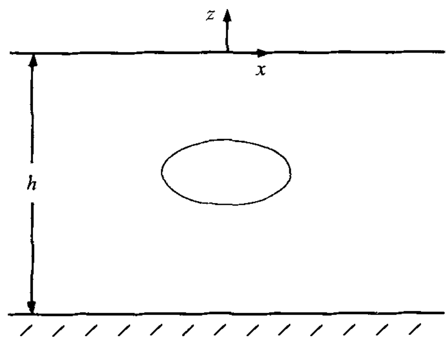

FIGURE 1. Definition sketch.

of motion. In two dimensions, the radiated motion may be decomposed into three possible modes: heave, which is motion parallel to the z-axis; sway, which is motion parallel to the x-axis; and roll, which is rotation about the y-axis. The purpose of this section is to express the scattering potential at each point in the fluid in terms of the heave and sway radiation potentials.

The scattering potential due to a wave of amplitude A and frequency ω incident on the body from the left is given by

$$\Phi_s = \mathrm{Re}\left[\frac{-\mathrm{i}gA}{\omega}\phi_s(x,z)\mathrm{e}^{-\mathrm{i}\omega t}\right], \tag{2.1}$$

where g is the acceleration due to gravity. The potential ϕ_s satisfies

$$\nabla^2\phi_s = 0 \quad \text{in the fluid}, \tag{2.2}$$

$$K\phi_s - \frac{\partial \phi_s}{\partial z} = 0 \quad \text{on} \quad z = 0, \tag{2.3}$$

where $K = \omega^2/g$,

$$\frac{\partial \phi_s}{\partial n} = 0 \qquad \text{on the body}, \tag{2.4}$$

$$\frac{\partial \phi_s}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{2.5}$$

and has the far-field behaviour

$$\phi_s \sim \frac{\cosh k(z+h)}{\cosh kh}\begin{cases} \mathrm{e}^{\mathrm{i}kx} + R^-\mathrm{e}^{-\mathrm{i}kx} & \text{as } x \to -\infty, \\ T^-\mathrm{e}^{\mathrm{i}kx} & \text{as } x \to \infty, \end{cases} \tag{2.6}$$

where k is the only positive root of the dispersion relation

$$K = k \tanh kh, \tag{2.7}$$

and R^- and T^- are the reflection and transmission coefficients associated with a wave incident from minus infinity.

The radiation potential due to a body oscillating in mode α with amplitude ξ_α is given by

$$\Phi_\alpha = \mathrm{Re}[-\mathrm{i}\omega\xi_\alpha\phi_\alpha(x,z)\mathrm{e}^{-\mathrm{i}\omega t}], \tag{2.8}$$

where ϕ_α satisfies Laplace's equation, the free surface condition (2.3) and the seabed condition (2.5) and

$$\frac{\partial \phi_\alpha}{\partial n} = n_\alpha \quad \text{on the body,} \tag{2.9}$$

where n_α is the component of the inward normal to the body in the direction of oscillation of mode α, ($\alpha = 2$ corresponds to heave and $\alpha = 3$ corresponds to sway). At large distances

$$\phi_\alpha \sim \frac{\cosh k(z+h)}{\cosh kh} \begin{cases} A_\alpha^- e^{-ikx} & \text{as } x \to -\infty, \\ A_\alpha^+ e^{ikx} & \text{as } x \to \infty. \end{cases} \tag{2.10}$$

In order to relate the scattering and radiation potentials, a combination of the radiation potentials is formed which satisfies the same boundary conditions on the body and which has the same far-field behaviour as the scattering potential. From (2.9), the normal derivatives of both the heave and sway potentials on the body are real and so any linear combination of $\phi_2 - \overline{\phi_2}$ and $\phi_3 - \overline{\phi_3}$, where overbar denotes complex conjugate, automatically satisfies the homogeneous body boundary condition given in (2.4). The two constants in the combination are chosen so that the resulting potential does not contain any incoming wave from plus infinity and so that the amplitude of the incoming wave from minus infinity is the same as that in the scattering potential. The result is that the potential

$$\psi = \frac{1}{\overline{D}}[\overline{A_3^+}(\phi_2 - \overline{\phi_2}) - \overline{A_2^+}(\phi_3 - \overline{\phi_3})] \tag{2.11}$$

is formed, where

$$D = A_2^+ A_3^- - A_2^- A_3^+. \tag{2.12}$$

Clearly this definition of ψ is only meaningful if $D \neq 0$. For a symmetric body, $A_2^+ = A_2^- = A_2$ say and $A_3^+ = -A_3^- = A_3$ say and so

$$D = -2A_3 A_2. \tag{2.13}$$

Thus, $D = 0$ for a symmetric body only when $A_2 = 0$ or $A_3 = 0$. Zeros in these coefficients may occur and they correspond to the situation in which a body oscillates in a particular mode at a certain frequency and produces no waves. In this case, the corresponding radiation potential is purely real and the ratio in (2.11) is indeterminate. Such zeros are usually associated with twin-hull vessels and, in particular, they were found by Wang & Wahab (1971) for pairs of cylinders oscillating in the free surface. The frequencies at which they occur are close to the natural frequencies of oscillation of the fluid trapped between the cylinders. These zeros occur at isolated frequencies and are associated with regions of rapid change in the phase of the radiated wave in the neighbourhood of such a frequency. In order to analyse the limiting behaviour of (2.11) near a zero of A_2 or A_3 knowledge of $\partial\phi_\alpha/\partial K$ at the relevant frequency is needed and this is not readily available. Thus for the remainder of this work attention is restricted to frequencies for which $D \neq 0$. The potential ψ formed in (2.11) satisfies Laplace's equation, the free surface boundary condition (2.3) and the seabed condition (2.5). It is also designed to have zero normal derivative on the body and to represent the scattering of a wave incident from minus infinity. Straightforward algebraic manipulation shows that ψ has the far-field behaviour

$$\psi \sim \frac{\cosh k(z+h)}{\cosh kh} \begin{cases} e^{ikx} + (1/\overline{D})(\overline{A_3^+}A_2^- - \overline{A_2^+}A_3^-)e^{-ikx} & \text{as } x \to -\infty, \\ (1/\overline{D})(\overline{A_3^+}A_2^+ - \overline{A_2^+}A_3^+)e^{ikx} & \text{as } x \to \infty. \end{cases} \tag{2.14}$$

A comparison of (2.6) and (2.14) demonstrates that ϕ_s may be written as

$$\phi_s = \psi + \chi, \tag{2.15}$$

where χ satisfies (2.2)–(2.5) and has the far-field behaviour

$$\chi \sim \frac{\cosh k(z+h)}{\cosh kh} \begin{cases} \chi^- \mathrm{e}^{-\mathrm{i}kx} & \text{as } x \to -\infty, \\ \chi^+ \mathrm{e}^{\mathrm{i}kx} & \text{as } x \to \infty, \end{cases} \tag{2.16}$$

where, from (2.6) and (2.14)

$$\chi^- = R^- - \frac{1}{\overline{D}}\left(\overline{A_3^+} A_2^- - \overline{A_2^+} A_3^-\right) \tag{2.17}$$

and

$$\chi^+ = T^- - \frac{1}{\overline{D}}\left(\overline{A_3^+} A_2^+ - \overline{A_2^+} A_3^+\right). \tag{2.18}$$

An application of Green's theorem to χ and $\overline{\chi}$ yields

$$\chi^- = \chi^+ = 0, \tag{2.19}$$

which, when applied to (2.17) and (2.18), gives the well-known reciprocity relations derived by Newman (1975). In particular, these relations reduce to

$$R = -\frac{1}{2}\left[\frac{A_2}{\overline{A_2}} + \frac{A_3}{\overline{A_3}}\right] \tag{2.20}$$

and

$$T = \frac{1}{2}\left[\frac{A_3}{\overline{A_3}} - \frac{A_2}{\overline{A_2}}\right], \tag{2.21}$$

for symmetric bodies. Thus, the potential χ satisfies homogeneous boundary conditions everywhere and produces no waves at infinity. Provided that the solution to the boundary value problem is unique,

$$\chi \equiv 0 \quad \text{in the fluid.} \tag{2.22}$$

Uniqueness has not been proved for all two-dimensional bodies and recent work of the author, McIver (1996), shows that for at least some classes of pairs of surface-piercing bodies, the potential is not unique at all frequencies. However, uniqueness has been proved for a wide class of geometries and in particular, John (1950) proved uniqueness for single surface-piercing non-bulbous bodies. His work was extended by Simon & Ursell (1984) to include a range of submerged and bulbous bodies which could be contained within certain pairs of lines emanating from the free surface. The solution for the submerged circular cylinder was the first solution for a submerged body which was proved to be unique by Ursell (1950). The potentials for a further large class of submerged and surface-piercing bodies were demonstrated to be unique by Hulme (1984) using a complicated vector identity due to Maz'ja (1978). The remainder of this work applies only at frequencies at which the potential is unique and in this case the scattering potential ϕ_s is equal to the potential ψ derived from the radiation potentials. In particular, for symmetric bodies this yields

$$\phi_s = \frac{1}{2\overline{A_3}}\left(\phi_3 - \overline{\phi_3}\right) - \frac{1}{2\overline{A_2}}\left(\phi_2 - \overline{\phi_2}\right). \tag{2.23}$$

By using matched asymptotic expansions and assuming that the fluid has infinite depth, it may be shown that this representation reduces to that given by the relative motion hypothesis at low frequencies (Newman 1977, §6.18), which states that near the body

$$\phi_s \sim e^{iKx+Kz} - iK\phi_3 - K\phi_2. \tag{2.24}$$

However, the analysis is non-trivial because to leading order the inner solutions for ϕ_2 and ϕ_3 are real and, from (2.23), it is necessary to derive the imaginary part of the higher-order solution before the equivalence of the formulae can be demonstrated. The advantage of the representation for ϕ_s in (2.23) is that it is valid for all frequencies and at all points in the fluid. It may be used, for example, to relate the scattering and radiation pressures at arbitrary points on the body. Calculation of the first-order exciting forces and the far-field scattered waves from this representation simply yields the familiar reciprocity relations, but using the formula to calculate higher-order forces yields some new results, which will be explored in §4.

The two-dimensional scattering potential has been expressed in terms of the heave and sway potentials but it could equally well have been related to the heave and roll potentials. The roll potential is given by $\mathrm{Re}[-i\omega\xi_4\phi_4(x,z)e^{-i\omega t}]$ where ϕ_4 satisfies Laplace's equation, the free surface condition (2.3), the seabed condition (2.5) and

$$\frac{\partial\phi_4}{\partial n} = zn_x - xn_z \qquad \text{on the body,} \tag{2.25}$$

where n_x and n_z are the components of the inward normal to the body in the x- and z-directions. From (2.25), if the body is symmetric, the roll potential is antisymmetric and so ϕ_4 satisfies the far-field behaviour in (2.10) where $A_4^+ = -A_4^- = A_4$ say. If the potential is unique then a similar analysis to that already performed in this section shows that

$$\frac{1}{\overline{A_4}}(\phi_4 - \overline{\phi_4}) = \frac{1}{\overline{A_3}}(\phi_3 - \overline{\phi_3}), \tag{2.26}$$

provided that $A_4 \neq 0$ and $A_3 \neq 0$.

Unfortunately, the corresponding analysis does not work directly in three dimensions because, in this case, the radiation potentials represent outgoing cylindrical waves at infinity. This means that the combination of potentials $\phi_\alpha - \overline{\phi_\alpha}$ represents the scattering of incoming cylindrical waves not the scattering of plane waves, which is described by the scattering potential. It is in fact unlikely that the three-dimensional scattering potential may be written in terms of only six radiation potentials. This may be demonstrated by considering the diffraction of waves by an axisymmetric body. In general, the scattering of a plane wave excites an infinite sum of azimuthal modes; however each radiation problem excites at most one azimuthal mode and so it is impossible to model the scattering potential in terms of only the six radiation potentials. It may, however, be possible to extend the ideas of Davis (1976) who used the fact that a plane wave is expressible as an infinite sum of incoming and outgoing cylindrical waves in order to relate far-field radiation and scattering quantities. This idea was developed by Linton (1991) who solved a series of radiation-type problems for a submerged sphere, each of which was forced by a velocity distribution on the sphere with a different angular variation. He related the far-field scattering potential to a combination of these radiation potentials and it may be possible to extend his work to write the scattering potential in terms of an infinite series of these radiation potentials at all points in the fluid. However, such a relationship would not be quite as useful as the two-dimensional result because further radiation-type problems would have to be solved in addition to the six standard ones.

3. Two scattering potentials

A similar analysis to that used in the previous section is employed here to relate the scattering potential due to a wave incident from one infinity to the scattering potential due to a wave incident from the other infinity. In particular, for a symmetric body this result relates the scattering potential at a point $(-x, z)$ to the same potential at the point (x, z).

The far-field behaviour of the scattering potential due to a wave incident from minus infinity is given by

$$\phi_s \sim \frac{\cosh k(z+h)}{\cosh kh}\begin{cases} e^{ikx} + R^- e^{-ikx} & \text{as } x \to -\infty, \\ T^- e^{ikx} & \text{as } x \to \infty, \end{cases} \tag{3.1}$$

and the corresponding behaviour of the potential due to a wave incident from plus infinity, $\Gamma_s(x, z)$, is given by

$$\Gamma_s \sim \frac{\cosh k(z+h)}{\cosh kh}\begin{cases} T^+ e^{-ikx} & \text{as } x \to -\infty, \\ e^{-ikx} + R^+ e^{ikx} & \text{as } x \to \infty, \end{cases} \tag{3.2}$$

where reciprocity relations exist which relate R^-, T^-, R^+ and T^+. The potential

$$\psi(x, z) = \frac{1}{\overline{T^-}}(\overline{\phi_s}(x, z) - \overline{R^-}\phi_s(x, z)) \tag{3.3}$$

is formed, under the assumption that $T^- \neq 0$. (Zeros in transmission coefficients may occur particularly with multiple bodies and the phenomenon is discussed by Evans (1974). However, for the purpose of this work the transmission coefficient is assumed to be non-zero.) Clearly, ψ satisfies Laplace's equation and the same boundary conditions on the free surface, body and seabed as ϕ_s and it is straightforward to verify that the far-field behaviour of ψ is given by

$$\psi \sim \frac{\cosh k(z+h)}{\cosh kh}\begin{cases} T^- e^{-ikx} & \text{as } x \to -\infty, \\ e^{-ikx} - (\overline{R^-}T^-/\overline{T^-})e^{ikx} & \text{as } x \to \infty, \end{cases} \tag{3.4}$$

which corresponds to the far-field behaviour for the potential due to a wave incident from plus infinity. If the potential is unique, a similar argument to that used in the previous section shows that $\psi(x, z) = \Gamma_s(x, z)$ and, as a by-product, produces the reciprocity relations

$$T^+ = T^- = T \tag{3.5}$$

say and

$$R^+\overline{T^-} = -\overline{R^-}T^-. \tag{3.6}$$

In particular, if the body is symmetric, $\Gamma_s(x, z) = \phi_s(-x, z)$ and $R^- = R^+ = R$ say, and so equating $\phi_s(-x, z)$ to the expression for $\psi(x, z)$ given by (3.3) yields

$$\overline{T}\phi_s(-x, z) = \overline{\phi_s}(x, z) - \overline{R}\phi_s(x, z). \tag{3.7}$$

Several interesting results may be deduced from this relation. First, putting $x = 0$ yields

$$\frac{\overline{\phi_s}(0, z)}{\phi_s(0, z)} = \overline{R} + \overline{T}, \tag{3.8}$$

for points outside the body, provided that $\phi_s(0, z) \neq 0$, i.e. the phase of the potential on the line $x = 0$ is everywhere the same and related to the reflection and transmission coefficients. Similarly, differentiating (3.7) with respect to x and putting $x = 0$ gives

$$\frac{\partial \overline{\phi_s}}{\partial x}(0,z) \Big/ \frac{\partial \phi_s}{\partial x}(0,z) = \overline{R} - \overline{T}, \tag{3.9}$$

provided that $\partial \phi_s / \partial x(0,z) \neq 0$, which means that the phase of the horizontal velocity on the line $x = 0$ is everywhere the same. Further results may be obtained from (3.7) at more general points in the fluid. For example, if x is replaced by $-x$ in (3.7) and the left-hand side of the new equation is multiplied by the right-hand side of (3.7) and vice versa, then the result

$$|\phi_s(x,z)|^2 - |\phi_s(-x,z)|^2 = \overline{R}[\phi_s^2(x,z) - \phi_s^2(-x,z)] \tag{3.10}$$

is obtained. As the left-hand side of (3.10) is clearly real, the right-hand side must also be real which means that provided $R \neq 0$,

$$\phi_s^2(x,z) - \phi_s^2(-x,z) = RA(x,z), \tag{3.11}$$

where $A(x,z)$ is a real function. These results are consistent with the numerical findings of McIver (1992) who observed that for a half-immersed circular cylinder of radius a

$$\phi_s^2(a,0) - \phi_s^2(-a,0) = -4R \tag{3.12}$$

and

$$|\phi_s(a,0)|^2 - |\phi_s(-a,0)|^2 = -4|R|^2, \tag{3.13}$$

although the result proved here does not explain the numerical factor of -4 which was observed. Unfortunately, there is not yet an explanation for this factor which was purely a numerical observation.

4. Discussion

The relationship between the scattering and radiation potentials derived in §2 may be used to determine the first-order exciting forces and far-field scattered waves in terms of radiation quantities and this yields the reciprocity relations detailed in Newman (1976). However, because the relationship is valid throughout the fluid, it may be used to calculate higher-order quantities such as the mean exciting forces or the forcing term for the second-order potential, in terms of the radiation potentials. To illustrate the use of the result beyond the calculation of first-order quantities, expressions for the vertical mean and double-frequency forces arising from the first-order scattering potential are derived here in terms of the vertical mean and double frequency forces arising from the radiation potentials. (The corresponding horizontal forces have already been expressed in terms of far-field first-order quantities by Maruo 1960 and McIver 1992.)

For convenience, the body is assumed to be symmetric and to be either totally submerged or to intersect the undisturbed free surface at right angles. Mei (1983) showed that the non-dimensional vertical mean force on such a fixed body may be written as

$$f_{ms} = -\frac{1}{4K} \int_{\overline{S_B}} |\nabla \phi_s|^2 n_z \, \mathrm{d}S, \tag{4.1}$$

where $\overline{S_B}$ is the mean wetted surface of the body and n_z is the component of the inward unit normal to the body in the z-direction. Similarly, the non-dimensional vertical double-frequency force on such a body due to the first-order potential has the complex amplitude

$$f_{ds} = \frac{1}{4K} \int_{\overline{S_B}} (\nabla\phi_s)^2 \, n_z \, \mathrm{d}S. \tag{4.2}$$

From (2.23) ϕ_s may be written as

$$\phi_s = \mathrm{i}\left[\frac{1}{\overline{A_3}}\mathrm{Im}[\phi_3] - \frac{1}{\overline{A_2}}\mathrm{Im}[\phi_2]\right] \tag{4.3}$$

and so

$$|\nabla\phi_s|^2 = \frac{(\mathrm{Im}[\nabla\phi_2])^2}{|A_2|^2} + \frac{(\mathrm{Im}[\nabla\phi_3])^2}{|A_3|^2} - \mathrm{Im}[\nabla\phi_2]\cdot\mathrm{Im}[\nabla\phi_3]\left[\frac{1}{A_2\overline{A_3}} + \frac{1}{\overline{A_2}A_3}\right]. \tag{4.4}$$

The body is symmetric and so the heave potential, ϕ_2, is symmetric and the sway potential, ϕ_3, is antisymmetric. Thus, $(\mathrm{Im}[\nabla\phi_2])^2$ and $(\mathrm{Im}[\nabla\phi_3])^2$ are symmetric and $\mathrm{Im}[\nabla\phi_2]\cdot\mathrm{Im}[\nabla\phi_3]$ is antisymmetric and so

$$f_{ms} = -\frac{1}{4K} \int_{\overline{S_B}} \left[\frac{(\mathrm{Im}[\nabla\phi_2])^2}{|A_2|^2} + \frac{(\mathrm{Im}[\nabla\phi_3])^2}{|A_3|^2}\right] n_z \, \mathrm{d}S. \tag{4.5}$$

Straightforward manipulation yields

$$(\mathrm{Im}[\nabla\phi_i])^2 = \tfrac{1}{2}(|\nabla\phi_i|^2 - \mathrm{Re}[(\nabla\phi_i)^2]), \tag{4.6}$$

and this may be substituted into (4.5) to give

$$f_{ms} = \frac{1}{2|A_2|^2}\,[f_{m2} + \mathrm{Re}[f_{d2}]] + \frac{1}{2|A_3|^2}\,[f_{m3} + \mathrm{Re}[f_{d3}]], \tag{4.7}$$

where

$$f_{mi} = -\frac{1}{4K} \int_{\overline{S_B}} |\nabla\phi_i|^2 \, n_z \, \mathrm{d}S \tag{4.8}$$

and

$$f_{di} = \frac{1}{4K} \int_{\overline{S_B}} (\nabla\phi_i)^2 \, n_z \, \mathrm{d}S \tag{4.9}$$

are the contributions to the vertical mean and double-frequency radiation forces which arise from the quadratic terms in Bernoulli's equation when the body moves in a single mode of motion. A similar analysis yields the non-dimensional vertical double-frequency exciting force due to the first-order potential in terms of the radiation forces as

$$f_{ds} = \frac{1}{2\overline{A_2}^2}\,[f_{m2} + \mathrm{Re}[f_{d2}]] + \frac{1}{2\overline{A_3}^2}\,[f_{m3} + \mathrm{Re}[f_{d3}]]. \tag{4.10}$$

The expressions (4.7) and (4.10) are verified for the half-immersed circular cylinder of radius a in fluid of infinite depth. Both the scattering and radiation second-order forces are computed directly using the multipole method described in Martin & Dixon (1983) with 32 multipole potentials and then the scattering forces are calculated indirectly from (4.7) and (4.10). The results obtained from the two methods are the same to at least five decimal places and the values of $|f_{ds}|$ agree with the results presented graphically by Wu & Eatock Taylor (1989). In order to assist with any future comparisons, the results are presented in table 1. The level of agreement between the two methods was obtained even when only one multipole potential was

Ka	$\lvert f_{ms}\rvert$	$\lvert f_{ds}\rvert$
0.05	0.08104	0.05164
0.1	0.15748	0.10868
0.2	0.29452	0.22808
0.4	0.42305	0.37561
0.6	0.37426	0.35473
0.8	0.29373	0.28675
1.0	0.23120	0.22867
1.2	0.18782	0.18685
1.4	0.15911	0.15871
1.6	0.14177	0.14160
1.8	0.13340	0.13332
2.0	0.13173	0.13168

TABLE 1. Calculations of f_{ms} and f_{ds} for a half-immersed cylinder

used and convergence of the numerical results had not occurred. In the author's experience, this commonly happens when numerically testing reciprocity relations.

5. Conclusion

In this work, an expression for the scattering potential for a two-dimensional body was derived, in terms of the heave and sway radiation potentials, at all points in the fluid. When integrated, this expression generates the well-known first-order reciprocity relations. However, by applying the formula pointwise in the fluid, other relations between higher-order scattering and radiation forces have been derived. In addition, it has been demonstrated that the scattering potentials due to waves incident from one infinity or the other may be written in terms of each other. For a symmetric body, this leads to a relationship between the scattering potential at pairs of points in the fluid which was exploited to yield information about the phase of the potential and horizontal velocity on $x = 0$.

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